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A UNIFIED THEORY OF PLASTIC BUCKLING OF COLUMNS AND PLATES

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SUMMARY

On the basis of modern plasticity considerations, a unified theory of plastic buckling applicable to both columns and plates has been developed. For uniform compression, the theory shows that long columns which bend without appreciable twisting require the tangent modulus and that long flanges which twist without appreciable bending require the secant modulus. Structures that both bend and twist when they buckle require a modulus which is a combination of the secant modulus and the tangent modulus.

INTRODUCTION

The calculation of the critical compressive stress of columns and of structures made up of plates is an important problem in aircraft design. Formulas for the critical compressive stress have been worked out for a multitude of cases of both columns and plates, but these formulas are accurate only if the buckling takes place within the elastic range of the material. In present-day designs, most buckling occurs above the elastic range. The usual method of handling this problem is to retain all the formulas derived for the elastic case, but to try to discover an effective, or reduced, modulus of elasticity which will give the correct result when inserted into these formulas.

Column buckling was the first structural problem to be studied in the plastic range. In the latter part of the nineteenth century, Engesser proposed use of the tangent modulus as the reduced modulus for columns. At almost the same time, in the belief that the column would be strengthened by unloading on the convex side, Considère suggested that the effective modulus should lie between the tangent modulus and Young's modulus. This concept was subsequently refined by Engesser and by Von Kármán (reference 1) and led to what is generally known as the "double modulus."

Experiments have shown, however, that the Von Kármán double modulus gives values that are too high for the column strength (reference 2) and that the correct modulus is probably the tangent modulus. Shanley (reference 3) has stated the situation compactly as follows: "If the tangent modulus is used directly in the Euler formula, the resulting critical load is somewhat lower than that given by the reduced modulus theory. This simpler formula, originally proposed by Engesser, is now widely used by engineers, since it gives values that agree very well with test data." Further careful tests by Shanley (reference 4) and also by Langley structures research laboratory have shown that the unloading on one side of the column, postulated by Von Kármán, does not occur at buckling and that the correct modulus for columns is actually the tangent modulus. This conclusion also has theoretical justification (references 3 and 4).

In the case of local or plate buckling, the reduced modulus is appreciably higher than the tangent modulus. Tests of the local buckling stress of aircraft-section columns have been made by Gerard (reference 5), who has suggested the use of the secant modulus for this type of buckling. Extensive tests in the Langley structures research laboratory on similar aircraft sections made and reported over a period of several years and summarized in reference 6 have also shown that the reduced modulus for plates is in the vicinity of the secant modulus. In particular, tests of long aluminum-alloy cruciform-section columns, designed to buckle by twisting without appreciable bending, have been made in a manner similar to that described for the aircraft-section columns in reference 6. The results have shown that the reduced modulus for pure twisting is very close to the secant modulus.

The present paper constitutes a theoretical investigation of the buckling of plates beyond the elastic range, which includes columns as a limiting case. Such an investigation requires a knowledge of the relations between the stress and strain components beyond the elastic range. These relations have not as yet been conclusively determined. A recent paper by Handelman and Prager (reference 7) based on one possible set of stress-strain relations led to results for the buckling of hinged flanges in sharp disagreement with test results obtained at the Langley structures research laboratory. Another set of stress-strain relations is generally accepted by the Russian investigators and has been applied by Ilyushin (reference 8) to the stress conditions in thin plates. These results form the foundation of the present paper, which assumes that in plates as well as in columns unloading during the early stages of buckling does not occur. On this basis, a unified theory of plastic buckling applicable to both column and local buckling has been developed. The results are presented in the following section.

RESULTS AND CONCLUSIONS

Ilyushin (reference 8) has treated the stability of plates stressed above the elastic limit with consideration of the three possible zones that might result from buckling: (1) a purely elastic zone, (2) a zone in which part of the material is in the elastic and part is in the plastic state—the "elasto-plastic" zone, and (3) a purely plastic zone in which all of the plate is stressed beyond the elastic limit. All three zones may exist simultaneously if the plate is not entirely in the plastic state before buckling or if the buckling is allowed to proceed beyond the initial stages.

If, however, the plate is uniformly loaded before buckling so that all parts of it are initially at the same point in the plastic range and if, in addition, buckling and increase in load are assumed to progress simultaneously, then the plate may be expected to remain in the purely plastic state in the early stages of buckling. This second assumption is in agreement with the corresponding condition that apparently holds for columns (reference 4).

Upon the assumption that the plate remains in the purely plastic state during buckling, Ilyushin's general relations for this state have been used to derive the differential equation of equilibrium of the plate under combined loads. Since critical stresses are generally simpler to compute from energy expressions than from a differential equation, the corresponding energy expressions were also found. These derivations are given in appendix A, together with applications to compressive buckling of various types of plates. A comparison with Ilyushin's treatment of the plastic-buckling problem is given in appendix B.

The results of most interest in the present analysis are given in the following table as values of a quantity η, the number by which the critical stress computed for the elastic case must be multiplied to give the critical stress for the plastic case.

Structure	η	Curve (See fig. 1)
Long flange, one unloaded edge simply supported	$\frac{E_{sec}}{E}$	A
Long flange, one unloaded edge clamped	$\frac{E_{sec}}{E}\left(0.330+0.670\sqrt{\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}}\right)$	B
Long plates, both unloaded edges simply supported	$\frac{E_{sec}}{E}\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}}\right)$	C
Long plate, both unloaded edges clamped	$\frac{E_{sec}}{E}\left(0.352+0.648\sqrt{\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}}\right)$	D
Short plate loaded as a column $\left(\frac{l}{b}\ll 1\right)$	$\frac{1}{4}\frac{E_{sec}}{E}+\frac{3}{4}\frac{E_{tan}}{E}$	E
Square plate loaded as a column $\left(\frac{l}{b}=1\right)$	$0.114\frac{E_{sec}}{E}+0.886\frac{E_{tan}}{E}$	F
Long column $\left(\frac{l}{b}\gg 1\right)$	$\frac{E_{tan}}{E}$	G

These values of η are plotted as curves A to G in figure 1 for extruded 24S–T aluminum alloy for which the compressive yield stress was 46 ksi. Similar curves for η could readily be prepared for any other material having a known stress-strain relationship.

The values of η given in the table were obtained by dividing the critical stress of the structure in the plastic region by the critical stress that would be obtained on the assumption of perfect elasticity. Since Poisson's ratio has been taken as one-half in both computations, errors from this cause will ordinarily be present in both critical stresses. Most of these errors will be eliminated, however, in the process of division to obtain η; and, consequently, the values of η given are believed to be nearly correct.

When plate-buckling stresses in the plastic range are to be computed, the experimental value of Poisson's ratio that applies as closely as possible to the stressed material, together with the appropriate value of η from this paper, should be used in the plate-buckling formula.

The highest value of η which is $\frac{E_{sec}}{E}$ can be realized only if there is negligible longitudinal bending (as with a long hinged flange which buckles by twisting). The lowest value of η which is $\frac{E_{tan}}{E}$ occurs when the longitudinal bending predominates over other types of distortion (as with a long column under Euler buckling). The theory implies that a change in the stress-strain curve caused by prestressing of the material would alter the value of η in the first case but not in the second, if the buckling stress is higher than the highest stress reached during the operation of prestressing. If, on the other hand, the buckling stress is lower than the highest stress reached during the operation of prestressing, then $\eta=1$ for each case.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *July 29, 1947.*

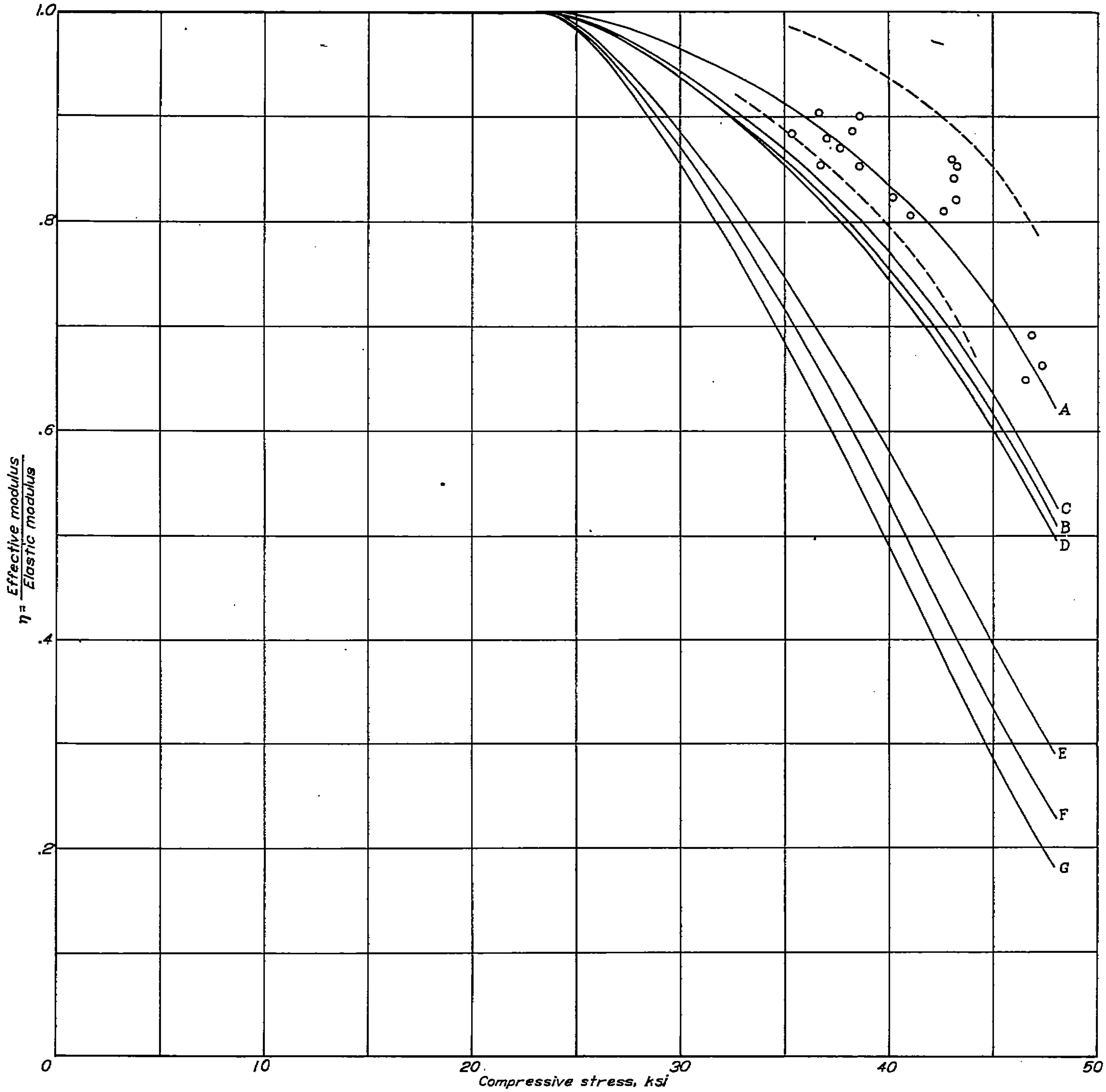

FIGURE 1.—Computed curves showing variation of η with stress for various structures of 24S-T aluminum alloy in compression. (Curves A to G are drawn for a material with a yield stress of 46 ksi. Circles are test data from cruciform sections and the scatter band enclosing curve A shows the limits of variation of the cruciform-section properties from 46 ksi.

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APPENDIX A

THEORETICAL DERIVATIONS

Definitions.—The intensities of stress and strain are defined in reference 8, respectively, as

$$\sigma_i = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau^2} \tag{1}$$

$$e_i = \frac{2}{\sqrt{3}}\sqrt{\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\epsilon_y + \frac{\gamma^2}{4}} \tag{2}$$

where

σ_x stress in the x-direction
ϵ_x strain in the x-direction
σ_y stress in the y-direction
ϵ_y strain in the y-direction
τ shear stress
γ shear strain

According to the fundamental hypothesis of the theory of plasticity, the intensity of stress σ_i is a uniquely defined, single-valued function of the intensity of strain e_i for any given material if σ_i increases in magnitude (loading condition). If σ_i decreases (unloading condition), the relation between σ_i and e_i becomes linear as in a purely elastic case.

In the equations of definition (1) and (2), the material is taken to be incompressible and Poisson's ratio$=\frac{1}{2}$. The stress-strain relations compatible with the equations of definition (1) and (2) are:

$$\left.\begin{aligned}
\epsilon_x &= \frac{\sigma_x - \frac{1}{2}\sigma_y}{E_{sec}} = \frac{S_x}{E_{sec}} \\
\epsilon_y &= \frac{\sigma_y - \frac{1}{2}\sigma_x}{E_{sec}} = \frac{S_y}{E_{sec}} \\
\gamma &= \frac{3\tau}{E_{sec}} \\
e_i &= \frac{\sigma_i}{E_{sec}}
\end{aligned}\right\} \tag{3}$$

These relations imply isotropy of the material.

Variations of strain and stress.—When buckling occurs, let ϵ_x, ϵ_y, and γ vary slightly from their values before buckling. The variations $\delta\epsilon_x$, $\delta\epsilon_y$, and $\delta\gamma$ will arise partly from the variations of middle-surface strains and partly from strains due to bending; thus,

$$\left.\begin{aligned}
\delta\epsilon_x &= \epsilon_1 - z\chi_1 \\
\delta\epsilon_y &= \epsilon_2 - z\chi_2 \\
\delta\gamma &= 2\epsilon_3 - 2z\chi_3
\end{aligned}\right\} \tag{4}$$

in which ϵ_1 and ϵ_2 are middle-surface strain variations and ϵ_3 is the middle-surface shear-strain variation, χ_1 and χ_2 are the changes in curvature and χ_3 is the change in twist, and z is the distance out from the middle surface of the plate.

The corresponding variations δS_x, δS_y, and $\delta\tau$ in S_x, S_y, and τ must be computed. From equations (3),

$$S_x = E_{sec}\epsilon_x$$

therefore

$$\delta S_x = E_{sec}\delta\epsilon_x + \epsilon_x\delta\left(\frac{\sigma_i}{e_i}\right)$$

$$= E_{sec}\delta\epsilon_x - \frac{\epsilon_x}{e_i}\left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right)\delta e_i \tag{5}$$

Now the variation of the work of the internal forces is

$$\sigma_i\delta e_i = \sigma_x\delta\epsilon_x + \sigma_y\delta\epsilon_y + \tau\,\delta\gamma$$

so that

$$\delta e_i = \frac{\sigma_x\delta\epsilon_x + \sigma_y\delta\epsilon_y + \tau\,\delta\gamma}{\sigma_i}$$

$$= \frac{\sigma_x\epsilon_1 + \sigma_y\epsilon_2 + 2\tau\epsilon_3 - z(\sigma_x\chi_1 + \sigma_y\chi_2 + 2\tau\chi_3)}{\sigma_i} \tag{6}$$

Substitution of this value of δe_i in equation (5) gives

$$\delta S_x = E_{sec}\delta\epsilon_x - \frac{\epsilon_x}{\sigma_i e_i}\left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right)[\sigma_x\epsilon_1 + \sigma_y\epsilon_2 + 2\tau\epsilon_3 - z(\sigma_x\chi_1 + \sigma_y\chi_2 + 2\tau\chi_3)]$$

Let the coordinate of the surface for which $\delta e_i = 0$ (the neutral surface) be $z = z_0$. The expression for z_0 is obtained by setting $\delta e_i = 0$ in equation (6);

$$z_0 = \frac{\sigma_x\epsilon_1 + \sigma_y\epsilon_2 + 2\tau\epsilon_3}{\sigma_x\chi_1 + \sigma_y\chi_2 + 2\tau\chi_3} \tag{7}$$

By introduction of this coordinate into the expression for δS_x and by recognition of $\frac{\sigma_i}{e_i}$ as E_{sec} and $\frac{d\sigma_i}{de_i}$ as E_{tan}

$$\delta S_x = E_{sec}(\epsilon_1 - z\chi_1) + \frac{\epsilon_x}{\sigma_i e_i}(E_{sec} - E_{tan})(\sigma_x\chi_1 + \sigma_y\chi_2 + 2\tau\chi_3)(z - z_0) \tag{8}$$

In a similar way it may be shown that

$$\delta S_y = E_{sec}(\epsilon_2 - z\chi_2) + \frac{\epsilon_y}{\sigma_i e_i}(E_{sec} - E_{tan})(\sigma_x\chi_1 + \sigma_y\chi_2 + 2\tau\chi_3)(z - z_0) \tag{9}$$

and

$$\delta\tau = \frac{2}{3}E_{sec}(\epsilon_3 - z\chi_3) + \frac{\gamma}{3\sigma_i e_i}(E_{sec} - E_{tan})(\sigma_x\chi_1 + \sigma_y\chi_2 + 2\tau\chi_3)(z - z_0) \tag{10}$$

Variations of forces and moments.—For the variations of the impressed forces T_x, T_y, and T_{xy} and the moments M_x, M_y, and M_{xy}

$$\left.\begin{aligned}\delta T_x&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\delta\sigma_x dz\\ \delta T_y&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\delta\sigma_y dz\\ \delta T_{xy}&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\delta\tau\, dz\\ \delta M_x&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\delta\sigma_x z\, dz\\ \delta M_y&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\delta\sigma_y z\, dz\\ \delta M_{xy}&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\delta\tau z\, dz\end{aligned}\right\} \tag{11}$$

where h is the thickness of the plate.

From equations (3), (8), and (9),

$$\begin{aligned}\delta M_x&=\frac{4}{3}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\delta S_x+\frac{1}{2}\,\delta S_y\right)z\,dz\\ &=\frac{4}{3}\left[E_{sec}\left(\epsilon_1+\frac{1}{2}\,\epsilon_2\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}z\,dz-E_{sec}\left(\chi_1+\frac{1}{2}\,\chi_2\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}z^2dz+\frac{\epsilon_x+\frac{1}{2}\,\epsilon_y}{\sigma_i e_i}\,(E_{sec}-E_{tan})\,(\sigma_x\chi_1+\sigma_y\chi_2+2\tau\chi_3)\int_{-\frac{h}{2}}^{\frac{h}{2}}(z^2-zz_0)dz\right]\\ &=\frac{4}{3}\,\frac{E_{sec}h^3}{12}\left[-\left(\chi_1+\frac{1}{2}\,\chi_2\right)+\frac{\epsilon_x+\frac{1}{2}\,\epsilon_y}{\sigma_i e_i}\left(1-\frac{E_{tan}}{E_{sec}}\right)(\sigma_x\chi_1+\sigma_y\chi_2+2\tau\chi_3)\right]\\ &=-D'\left\{\left[1-\frac{3}{4}\left(\frac{\sigma_x}{\sigma_i}\right)^2\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\chi_1+\frac{1}{2}\left[1-\frac{3}{2}\,\frac{\sigma_x\sigma_y}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\chi_2-\frac{3}{2}\,\frac{\sigma_x\tau}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\chi_3\right\}\end{aligned} \tag{12}$$

where

$$D'=\frac{E_{sec}h^3}{9}$$

Similarly,

$$\begin{aligned}\delta M_y&=\frac{4}{3}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\delta S_y+\frac{1}{2}\,\delta S_x\right)z\,dz\\ &=-D'\left\{\left[1-\frac{3}{4}\left(\frac{\sigma_y}{\sigma_i}\right)^2\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\chi_2+\frac{1}{2}\left[1-\frac{3}{2}\,\frac{\sigma_x\sigma_y}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\chi_1-\frac{3}{2}\,\frac{\sigma_y\tau}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\chi_3\right\}\end{aligned} \tag{13}$$

$$\begin{aligned}\delta M_{xy}&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\delta\tau z\,dz\\ &=-\frac{D'}{2}\left\{\left[1-\frac{3\tau^2}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\chi_3-\frac{3}{2}\left(\frac{\sigma_x\tau}{\sigma_i^2}\,\chi_1+\frac{\sigma_y\tau}{\sigma_i^2}\,\chi_2\right)\left(1-\frac{E_{tan}}{E_{sec}}\right)\right\}\end{aligned} \tag{14}$$

In these expressions, the integrations of δS_x, δS_y, and $\delta\tau$ in the plastic region have been taken over the entire thickness of the plate, with the implication that no part of the plate is being unloaded.

Equation of equilibrium.—If $w\ (x, y)$ is the bending deflection of the plate at buckling, and if no external moments are applied to the plate, then the equation of equilibrium of an element of the plate may be written

$$\frac{\partial^2(\delta M_x)}{\partial x^2}+2\frac{\partial^2(\delta M_{xy})}{\partial x\partial y}+\frac{\partial^2(\delta M_y)}{\partial y^2}=h\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+\sigma_y\frac{\partial^2 w}{\partial y^2}+2\tau\frac{\partial^2 w}{\partial x\partial y}\right) \tag{15}$$

in which the impressed forces $\sigma_x h$, $\sigma_y h$, and τh are considered as given (σ_x and σ_y are positive for compression). In terms of w, the changes in curvatures are

$$\chi_1=\frac{\partial^2 w}{\partial x^2} \tag{16a}$$

and

$$\chi_2=\frac{\partial^2 w}{\partial y^2} \tag{16b}$$

The change in twist is

$$\chi_3=\frac{\partial^2 w}{\partial x\partial y} \tag{16c}$$

When the values of δM_x, δM_y, and δM_{xy} in equations (12), (13), and (14), respectively, are differentiated as required by equation (15) and substituted in that equation, the general differential equation of equilibrium for a plate in the plastic state is obtained as follows:

$$\left[1-\frac{3}{4}\left(\frac{\sigma_x}{\sigma_i}\right)^2\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\frac{\partial^4 w}{\partial x^4}-3\frac{\sigma_x\tau}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\frac{\partial^4 w}{\partial x^3\partial y}+2\left[1-\frac{3}{4}\frac{\sigma_x\sigma_y+2\tau^2}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\frac{\partial^4 w}{\partial x^2\partial y^2}-$$
$$3\frac{\sigma_y\tau}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)\frac{\partial^4 w}{\partial x\partial y^3}+\left[1-\frac{3}{4}\left(\frac{\sigma_y}{\sigma_i}\right)^2\left(1-\frac{E_{tan}}{E_{sec}}\right)\right]\frac{\partial^4 w}{\partial y^4}=-\frac{h}{D'}\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+\sigma_y\frac{\partial^2 w}{\partial y^2}+2\tau\frac{\partial^2 w}{\partial x\partial y}\right) \tag{17}$$

In the elastic range, equation (17) reduces to the usual form

$$\nabla^4 w=-\frac{h}{D}\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+\sigma_y\frac{\partial^2 w}{\partial y^2}+2\tau\frac{\partial^2 w}{\partial x\partial y}\right)$$

where

$$D=\frac{Eh^3}{9}$$

Energy integrals.—Equation (17) is the Euler equation that results from a minimization of the integral

$$\iint\left(\frac{D'}{2}\left\{C_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2-C_2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y}+C_3\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]-C_4\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w}{\partial y^2}+C_5\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right\}-\right.$$
$$\left.\frac{h\sigma_i}{2}\left[\frac{\sigma_x}{\sigma_i}\left(\frac{\partial w}{\partial x}\right)^2+\frac{2\tau}{\sigma_i}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\sigma_y}{\sigma_i}\left(\frac{\partial w}{\partial y}\right)^2\right]\right)dx\ dy \tag{18}$$

which represents the difference between the strain energy in the plate and the work done on the plate by the external forces. The coefficients in this integral are:

In the plastic region	*In the elastic region*
$C_1=1-\frac{3}{4}\left(\frac{\sigma_x}{\sigma_i}\right)^2\left(1-\frac{E_{tan}}{E_{sec}}\right)$	$C_1=1$
$C_2=3\frac{\sigma_x\tau}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)$	$C_2=0$
$C_3=1-\frac{3}{4}\frac{\sigma_x\sigma_y+2\tau^2}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)$	$C_3=1$
$C_4=3\frac{\sigma_y\tau}{\sigma_i^2}\left(1-\frac{E_{tan}}{E_{sec}}\right)$	$C_4=0$
$C_5=1-\frac{3}{4}\left(\frac{\sigma_y}{\sigma_i}\right)^2\left(1-\frac{E_{tan}}{E_{sec}}\right)$	$C_5=1$

If there is a restraint of magnitude ϵ along one longitudinal edge of the plate, the strain energy in this restraint itself is taken to be

$$\frac{\epsilon}{2}\frac{D'}{b}\int\left[\left(\frac{\partial w}{\partial y}\right)_{y=y_0}\right]^2 dx \tag{19}$$

if y_0 is the edge coordinate. (See reference 9 for form of expression.) In expression (19), the stiffness D' is assumed to be the same as that in equation (12). If restraints are present along two edges, there will be two terms similar to expression (19). These terms may be added to integral (18) as additional strain energy.

Critical stress in plastic region.—If the integral (18), supplemented if necessary by additional terms of the form of expression (19), is set equal to zero and the resulting equation solved for σ_i, the critical-stress intensity in the plastic region $(\sigma_i)_{pl}$ is

$$(\sigma_i)_{pl}=\frac{D'}{h}\frac{\iint\left\{C_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2-C_2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y}+C_3\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]-C_4\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 w}{\partial y^2}+C_5\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right\}dxdy+\frac{\epsilon}{b}\int\left[\left(\frac{\partial w}{\partial y}\right)_{y=y_0}\right]^2 dx}{\iint\left[\frac{\sigma_x}{\sigma_i}\left(\frac{\partial w}{\partial x}\right)^2+2\frac{\tau}{\sigma_i}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\sigma_y}{\sigma_i}\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy} \tag{20}$$

in which the values of the C's in the plastic range are used. This expression for the critical-stress intensity may be minimized as with the corresponding elastic case.

If the values of the C's in the elastic region are used in formula (20), the critical-stress intensity in the elastic region $(\sigma_i)_{el}$ is as follows:

$$(\sigma_i)_{el}=\frac{D}{h}\frac{\iint\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]dx\,dy+\frac{\epsilon}{b}\int\left[\left(\frac{\partial w}{\partial y}\right)_{y=y_0}\right]^2 dx}{\iint\left[\frac{\sigma_x}{\sigma_i}\left(\frac{\partial w}{\partial x}\right)^2+2\frac{\tau}{\sigma_i}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\sigma_y}{\sigma_i}\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy} \tag{21}$$

Expression for η.—A quantity η is defined as

$$\eta=\frac{(\sigma_i)_{pl}}{(\sigma_i)_{el}} \tag{22}$$

This quantity is a direct measure of the effectiveness of plasticity in reducing the critical stress of a structure, and its computation in terms of the constants of the stress-strain curve represents the solution of the problem of plastic buckling.

Application to plates compressed in the x-direction.—The theory will now be applied to flat, rectangular plates uniformly compressed in the x-direction. Values of η will be computed for the following cases:

I. Long plates with one free edge (flanges), the other edge being either hinged or clamped
II. Long plates with both edges either hinged or clamped
III. Plates with two free edges (columns)

When $\sigma_y=\tau=0$, $\sigma_i=\sigma_x$ and the plasticity coefficients reduce to

$$C_1=\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}$$

$$C_2=C_4=0$$

$$C_3=C_5=1$$

The differential equation of equilibrium, equation (17), then becomes

$$C_1\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=-\frac{h\sigma_x}{D'}\frac{\partial^2 w}{\partial x^2} \tag{23}$$

and the corresponding energy expression (20) for the critical stress in the plastic range becomes

$$(\sigma_x)_{pl}=\frac{D'}{h}\frac{\iint\left[C_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]dx\,dy+\frac{\epsilon}{b}\int\left[\left(\frac{\partial w}{\partial y}\right)_{y=y_0}\right]^2 dx}{\iint\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy} \tag{24}$$

Case I: Elastically restrained flange

If $y=0$ is the elastically restrained edge of the flange and $y=b$ is the free edge, a deflection surface known to be good in the elastic range and presumably satisfactory also beyond this range is (reference 9)

$$w=\left\{\frac{y}{b}+\frac{\epsilon}{2a_3}\left[\left(\frac{y}{b}\right)^5+a_1\left(\frac{y}{b}\right)^4+a_2\left(\frac{y}{b}\right)^3+a_3\left(\frac{y}{b}\right)^2\right]\right\}\cos\frac{\pi x}{\lambda}$$

where

$$a_1=-4.963$$
$$a_2=9.852$$
$$a_3=-9.778$$

and ϵ is the magnitude of the elastic restraint. Substitution of this expression for w in equation (24) gives

$$(\sigma_x)_{pl}=\frac{2}{\pi^2}\frac{\frac{1}{2}+\frac{\epsilon}{2}\left(c_2-\frac{1}{2}c_3\right)+\frac{\epsilon^2}{4}\left[c_6-\frac{1}{2}c_7+\frac{c_5}{2\left(\frac{\pi b}{\lambda}\right)^2}\right]+\left(\frac{1}{6}+\frac{c_1\epsilon}{4}+\frac{c_4\epsilon^2}{8}\right)\left(\frac{\pi b}{\lambda}\right)^2C_1+\frac{\epsilon}{2\left(\frac{\pi b}{\lambda}\right)^2}}{\frac{1}{3}+\frac{c_8\epsilon}{2a_3}+\frac{c_9\epsilon^2}{4a_3^2}}\frac{\pi^2D'}{b^2h}$$

where

$c_1=0.23694$	$c_4=0.04286$	$c_7=0.19736$
$c_2=0.79546$	$c_5=0.56712$	$c_8=-2.3168$
$c_3=0.89395$	$c_6=0.17564$	$c_9=4.0982$

In order to find the minimum value of $(\sigma_x)_{pl}$,

$$\frac{\partial(\sigma_x)_{pl}}{\partial\left(\frac{\pi b}{\lambda}\right)^2}=0$$

which gives

$$\left(\frac{\pi b}{\lambda}\right)^2=\sqrt{\frac{\frac{\epsilon}{2}\left(1+\frac{c_5\epsilon}{4}\right)}{\left(\frac{1}{6}+\frac{c_1\epsilon}{4}+\frac{c_4\epsilon^2}{8}\right)C_1}}$$

The minimum value of $(\sigma_x)_{pl}$ is therefore

$$(\sigma_x)_{pl}=\frac{2}{\pi^2}\frac{\frac{1}{2}+\frac{\epsilon}{2}\left(c_2-\frac{1}{2}c_3\right)+\frac{\epsilon^2}{4}\left(c_6-\frac{1}{2}c_7\right)+2\sqrt{C_1}\sqrt{\frac{\epsilon}{2}\left(1+\frac{c_5\epsilon}{4}\right)\left(\frac{1}{6}+\frac{c_1\epsilon}{4}+\frac{c_4\epsilon^2}{8}\right)}}{\frac{1}{3}+\frac{c_8\epsilon}{2a_3}+\frac{c_9\epsilon^2}{4a_3^2}}\frac{\pi^2D'}{b^2h}$$

For the elastic case, the same expression is obtained from equation (21) with $C_1=1$ and D' replaced by D. From equation (22), therefore,

$$\eta=\frac{E_{sec}}{E}\frac{\frac{1}{2}+\frac{\epsilon}{2}\left(c_2-\frac{1}{2}c_3\right)+\frac{\epsilon^2}{4}\left(c_6-\frac{1}{2}c_7\right)+2\sqrt{C_1}\sqrt{\frac{\epsilon}{2}\left(1+\frac{c_5\epsilon}{4}\right)\left(\frac{1}{6}+\frac{c_1\epsilon}{4}+\frac{c_4\epsilon^2}{8}\right)}}{\frac{1}{2}+\frac{\epsilon}{2}\left(c_2-\frac{1}{2}c_3\right)+\frac{\epsilon^2}{4}\left(c_6-\frac{1}{2}c_7\right)+2\sqrt{\frac{\epsilon}{2}\left(1+\frac{c_5\epsilon}{4}\right)\left(\frac{1}{6}+\frac{c_1\epsilon}{4}+\frac{c_4\epsilon^2}{8}\right)}} \tag{25}$$

(a) If the edge $y=0$ is hinged, $\epsilon=0$ and, from equation (25),

$$\eta=\frac{E_{sec}}{E} \tag{26}$$

This value, as a function of stress, is plotted as curve A for 24S-T aluminum alloy in figure 1. The individual points represent the NACA tests of the buckling of cruciform-section columns for which the condition $\epsilon=0$ is fulfilled.

(b) If the edge $y=0$ is clamped, $\epsilon=\infty$ and, from equation (25),

$$\eta=\frac{E_{sec}}{E}\frac{\frac{1}{4}\left(c_6-\frac{1}{2}c_7\right)+2\sqrt{C_1}\frac{\sqrt{c_4c_5}}{8}}{\frac{1}{4}\left(c_6-\frac{1}{2}c_7\right)+2\frac{\sqrt{c_4c_5}}{8}}$$

or

$$\eta=\frac{E_{sec}}{E}\left(0.330+0.670\sqrt{\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}}\right) \tag{27}$$

This value of η is plotted as curve B for 24S-T aluminum alloy in figure 1.

Case II: Plate elastically restrained along two unloaded edges

If $y=\pm\frac{b}{2}$ are the immovable unloaded edges which are elastically restrained against rotation by restraints of magnitude ϵ, a satisfactory deflection surface is known to be (reference 10)

$$w=\left[\frac{\pi\epsilon}{2}\left(\frac{y^2}{b^2}-\frac{1}{4}\right)+\left(1+\frac{\epsilon}{2}\right)\cos\frac{\pi y}{b}\right]\cos\frac{\pi x}{\lambda}$$

Substitution of this expression for w in equation (24) gives

$$(\sigma_x)_{pl}=\left[C_1\left(\frac{b}{\lambda}\right)^2+f_1(\epsilon)\left(\frac{\lambda}{b}\right)^2+f_2(\epsilon)\right]\frac{\pi^2D'}{b^2h}$$

where

$$f_1(\epsilon)=\frac{0.0237\epsilon^2+0.297\epsilon+\frac{1}{2}}{0.00461\epsilon^2+0.0947\epsilon+\frac{1}{2}}$$

and

$$f_2(\epsilon)=\frac{0.0114\epsilon^2+0.1894\epsilon+1}{0.00461\epsilon^2+0.0947\epsilon+\frac{1}{2}}$$

In order to find the minimum value of $(\sigma_x)_{pl}$

$$\frac{\partial(\sigma_x)_{pl}}{\partial\left(\frac{b}{\lambda}\right)^2}=0$$

which gives

$$\left(\frac{b}{\lambda}\right)^2=\sqrt{\frac{f_1(\epsilon)}{C_1}}$$

The minimum value of $(\sigma_x)_{pl}$ is therefore

$$(\sigma_x)_{pl}=[2\sqrt{C_1}\sqrt{f_1(\epsilon)}+f_2(\epsilon)]\frac{\pi^2D'}{b^2h}$$

For the elastic case, the same expression is obtained from equation (21) with $C_1=1$ and D' replaced by D. From equation (22), therefore,

$$\eta=\frac{E_{sec}}{E}\frac{2\sqrt{C_1}\sqrt{f_1(\epsilon)}+f_2(\epsilon)}{2\sqrt{f_1(\epsilon)}+f_2(\epsilon)} \tag{28}$$

(a) If the edges $y=\pm\frac{b}{2}$ are hinged, $\epsilon=0$, $f_1(\epsilon)=1$, $f_2(\epsilon)=2$, and, from equation (28),

$$\eta=\frac{E_{sec}}{E}\frac{1+\sqrt{C_1}}{2}=\frac{E_{sec}}{E}\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}}\right) \tag{29}$$

This value of η is plotted as curve C for 24S–T aluminum alloy in figure 1.

(b) If the edges $y=\pm\frac{b}{2}$ are clamped, $\epsilon=\infty$, $f_1(\epsilon)=5.15$, $f_2(\epsilon)=2.46$, and, from equation (28),

$$\eta=\frac{E_{sec}}{E}\frac{2.46+4.52\sqrt{C_1}}{6.98}$$

$$=\frac{E_{sec}}{E}\left(0.353+0.647\sqrt{\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}}\right) \tag{30}$$

This value of η is plotted as curve D for 24S–T aluminum alloy in figure 1.

Case III: Plastic buckling of columns

For the discussion of the plastic buckling of columns, it is convenient to revert to the differential equation (23). The plate, when loaded as a column, has two free edges described by the conditions

$$\left(\frac{\partial^2w}{\partial y^2}+\frac{1}{2}\frac{\partial^2w}{\partial x^2}\right)_{y=\pm\frac{b}{2}}=0$$

$$\left(\frac{\partial^3w}{\partial y^3}+\frac{3}{2}\frac{\partial^3w}{\partial x^2\partial y}\right)_{y=\pm\frac{b}{2}}=0$$

A solution of equation (23) which identically satisfies the first condition is

$$w=\left(q\cos\frac{\beta}{2}\cosh\frac{\alpha y}{b}+p\cosh\frac{\alpha}{2}\cos\frac{\beta y}{b}\right)\cos\frac{\pi x}{l}$$

where

$$\alpha=\pi\sqrt{\frac{b}{l}}\sqrt{\frac{b}{l}+\sqrt{k+\left(\frac{b}{l}\right)^2(1-C_1)}}$$

$$\beta=\pi\sqrt{\frac{b}{l}}\sqrt{-\frac{b}{l}+\sqrt{k+\left(\frac{b}{l}\right)^2(1-C_1)}}$$

$$(\sigma_x)_{pl}=k\frac{\pi^2D'}{b^2h}$$

$$p=\alpha^2-\frac{1}{2}\left(\frac{\pi b}{l}\right)^2$$

$$q=\beta^2+\frac{1}{2}\left(\frac{\pi b}{l}\right)^2$$

$$C_1=\frac{1}{4}+\frac{3}{4}\frac{E_{tan}}{E_{sec}}$$

In order that this solution also satisfy the second condition at the free edges, it is required that

$$\alpha^2q\left[p-\left(\frac{\pi b}{l}\right)^2\right]\frac{\tanh\frac{\alpha}{2}}{\frac{\alpha}{2}}+\beta^2p\left[q+\left(\frac{\pi b}{l}\right)^2\right]\frac{\tan\frac{\beta}{2}}{\frac{\beta}{2}}=0 \tag{31}$$

which is the buckling criterion for the plate when loaded as a column. Let

$$k+\left(\frac{b}{l}\right)^2(1-C_1)=\left(\frac{b}{l}\right)^2(1-\xi^2) \tag{32}$$

where ξ^2 is a quantity to be determined for three individual cases. By use of equation (32)

$$\alpha^2=\left(\frac{\pi b}{l}\right)^2(1+\sqrt{1-\xi^2})$$

$$\beta^2=\left(\frac{\pi b}{l}\right)^2(-1+\sqrt{1-\xi^2})$$

$$p=\left(\frac{\pi b}{l}\right)^2\left(\frac{1}{2}+\sqrt{1-\xi^2}\right)$$

$$q=\left(\frac{\pi b}{l}\right)^2\left(-\frac{1}{2}+\sqrt{1-\xi^2}\right)$$

$$p-q=\left(\frac{\pi b}{l}\right)^2$$

and the buckling criterion given in equation (31) becomes

$$\alpha^2 q^2 \frac{\tanh\frac{\alpha}{2}}{\frac{\alpha}{2}}+\beta^2 p^2 \frac{\tan\frac{\beta}{2}}{\frac{\beta}{2}}=0$$

or

$$\left[\xi^2+\left(\frac{1}{4}-\xi^2\right)(1+\sqrt{1-\xi^2})\right]\frac{\tanh\frac{\alpha}{2}}{\frac{\alpha}{2}}-$$

$$\left[\xi^2+\left(\frac{1}{4}-\xi^2\right)(1-\sqrt{1-\xi^2})\right]\frac{\tan\frac{\beta}{2}}{\frac{\beta}{2}}=0 \tag{33}$$

From equation (32), $k=\left(\frac{b}{l}\right)^2(C_1-\xi^2)$; and thus the critical stress in the plastic range is

$$(\sigma_x)_{pl}=\frac{\pi^2 E_{sec}(C_1-\xi^2)}{\frac{3}{4}\left(\frac{l}{\rho}\right)^2}$$

where

$$\rho=\frac{h}{\sqrt{12}}$$

The corresponding critical stress in the elastic range is

$$(\sigma_x)_{el}=\frac{\pi^2 E(1-\xi^2)}{\frac{3}{4}\left(\frac{l}{\rho}\right)^2}$$

The reduction factor η is obtained from formula (22) as

$$\eta=\frac{\left(\frac{1}{4}-\xi^2\right)\frac{E_{sec}}{E}+\frac{3}{4}\frac{E_{tan}}{E}}{1-\xi^2} \tag{34}$$

(a) In order to investigate the case of short columns, let ξ approach zero. Then, by definition of β,

$$\beta\rightarrow 0$$

and

$$\frac{\tan\frac{\beta}{2}}{\frac{\beta}{2}}\rightarrow 1$$

In addition,

$$\xi^2+\left(\frac{1}{4}-\xi^2\right)\left(1-\sqrt{1-\xi^2}\right)\rightarrow 0$$

The buckling criterion given in equation (33) therefore reduces to

$$\left[\xi^2+\left(\frac{1}{4}-\xi^2\right)\left(1+\sqrt{1-\xi^2}\right)\right]\frac{\tanh\frac{\alpha}{2}}{\frac{\alpha}{2}}=0$$

The expression in the brackets approaches $\frac{1}{2}$ as $\xi\rightarrow 0$. In order to satisfy the buckling criterion, therefore,

$$\frac{\tanh\frac{\alpha}{2}}{\frac{\alpha}{2}}\rightarrow 0$$

which can be realized only if α is large; that is, if b/l is large. For short columns, therefore,

$$\xi\approx 0$$

and, from equation (34),

$$\eta=\frac{1}{4}\frac{E_{sec}}{E}+\frac{3}{4}\frac{E_{tan}}{E} \tag{35}$$

This value of η is plotted as curve E for 24S–T aluminum alloy in figure 1.

(b) For a square plate, $\frac{b}{l}=1$, $\alpha=\pi\sqrt{1+\sqrt{1-\xi^2}}$, $\beta=i\pi\sqrt{1-\sqrt{1-\xi^2}}$, and the buckling criterion given in equation (33) becomes

$$\left[\xi^2+\left(\frac{1}{4}-\xi^2\right)\left(1+\sqrt{1-\xi^2}\right)\right]\frac{\tanh\left(\frac{\pi}{2}\sqrt{1+\sqrt{1-\xi^2}}\right)}{\frac{\pi}{2}\sqrt{1+\sqrt{1-\xi^2}}}-$$

$$\left[\xi^2+\left(\frac{1}{4}-\xi^2\right)\left(1-\sqrt{1-\xi^2}\right)\right]\frac{\tanh\left(\frac{\pi}{2}\sqrt{1-\sqrt{1-\xi^2}}\right)}{\frac{\pi}{2}\sqrt{1-\sqrt{1-\xi^2}}}=0$$

which is satisfied by $\xi^2=0.15375$. From equation (34),

$$\eta=0.114\frac{E_{sec}}{E}+0.886\frac{E_{tan}}{E} \tag{36}$$

This value of η is plotted as curve F for 24S–T aluminum alloy in figure 1.

(c) For long columns, α and β become so small that

$$\frac{\tanh\frac{\alpha}{2}}{\frac{\alpha}{2}}\approx\frac{\tan\frac{\beta}{2}}{\frac{\beta}{2}}\approx 1$$

and the buckling criterion of equation (33) reduces to

$$\left[\xi^2+\left(\frac{1}{4}-\xi^2\right)\left(1+\sqrt{1-\xi^2}\right)\right]-\left[\xi^2+\left(\frac{1}{4}-\xi^2\right)\left(1-\sqrt{1-\xi^2}\right)\right]=0$$

which is satisfied by $\xi^2=\frac{1}{4}$. From equation (34),

$$\eta=\frac{E_{tan}}{E} \tag{37}$$

This value of η agrees with the experimental results of references 2 and 4 and is plotted as curve G for 24S–T aluminum alloy in figure 1.

APPENDIX B

COMPARISON WITH ILYUSHIN'S STABILITY CALCULATIONS

The basic difference between Ilyushin's solution of the plastic-buckling problem and that given in this paper is that Ilyushin considers the plate to unload on one face as it buckles. The unloading process results in the creation of an elastic-plastic zone in the plate, and different equations from those that apply when the plate remains plastic during the buckling process are required for this zone.

The differential equation for the buckling of a rectangular plate when buckling is accompanied by unloading is given by Ilyushin as equation (3.43) of reference 8. For simple compression in the x-direction this equation is of the same form as equation (23) of the present paper, but with the following different constants: D is used instead of D' and

$$k=1-\lambda\zeta^2(3-2\zeta)$$

is used instead of $\frac{E_{tan}}{E_{sec}}$. In the formula for k, from equation (3.1) of reference 8,

$$\zeta=\frac{1-\sqrt{1-\lambda}}{\lambda}$$

and, from equation (1.22) of reference 8,

$$\lambda=1-\frac{E_{tan}}{E}$$

When the values of ζ and λ are inserted into the expression for k

$$k=\frac{E_{tan}}{E}\left(2\,\frac{1-\sqrt{\frac{E_{tan}}{E}}}{1-\frac{E_{tan}}{E}}\right)^2$$

Computation shows that k is always larger than $\frac{E_{tan}}{E_{sec}}$, so that the use of k in place of $\frac{E_{tan}}{E_{sec}}$ will result in appreciably higher values of η than those given in the present paper. Since Ilyushin uses the elastic value D, there is no possibility of the solution yielding a secant modulus. Curves A to G in figure 1, if computed from Ilyushin's equation (3.43), would start with a horizontal line at unity for curve A (Young's modulus) and end with curve G expressing the Kármán double modulus which is appreciably higher than the tangent modulus of this paper. If D' were substituted for D in Ilyushin's equation (3.43), curve A would then represent the secant modulus as it does in the present paper, but curve G would still remain the Kármán double modulus. Therefore, when the unloading of the plate during the buckling process is considered, results are obtained which are not confirmed by experiment.

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